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HARMONICITY OF SECTIONS OF SPHERE BUNDLES

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Abstract. We consider the energy functional on the space of sections of a sphere bundle over a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ equipped with the Sasaki metric and we discuss the characterising condition for critical points. Likewise, we provide a useful method for computing the tension field in some particular situations. Such a method is shown to be adequate for many tensor fields defined on manifolds M equipped with a G-structure compatible with $\langle\cdot,\cdot\rangle$. This leads to the construction of a lot of new examples of differential forms which are harmonic sections or determine a harmonic map from $(M, \langle\cdot,\cdot\rangle)$ into its sphere bundle.

Keywords and phrases: energy of sections, harmonic section, harmonic map, G-structure, intrinsic torsion, minimal connection, almost Hermitian manifold, almost contact metric manifold, Riemannian curvature

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Contents

1. Introduction

The energy of a map between Riemannian manifolds is a functional which has been widely studied by diverse authors [9, 10, 31]. Critical points for the energy functional are called *harmonic maps* and have been characterised by Eells and Sampson [11] as maps whose *tension field* vanishes.

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For a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$, we denote by $(T_1M, \langle\cdot,\cdot\rangle^S)$ its unit tangent bundle equipped with the Sasaki metric $\langle\cdot,\cdot\rangle^S$ [1]. Thinking of unit vector fields as sections $M \to T_1M$, if M is compact and oriented, one can consider the energy functional as defined on the set $\mathfrak{X}_1(M)$ of unit vector fields. Critical points for this functional give rise to the notion of *harmonic unit vector field*. The condition characterising harmonic unit vector fields has been obtained by Wiegmink [33] (see also Wood's paper [34]). It is interesting to note that harmonic unit vector fields are not necessarily critical points for the energy functional on the space of all maps $M \to T_1M$. A harmonic unit vector field will also be harmonic map if and only if certain condition involving the curvature of M is satisfied.

In this paper we consider the energy functional defined on the space of sections of Riemannian vector bundles $\mathbb{E} \to M$, equipped with a metric which generalizes the Sasaki metric. In [15], Gil-Medrano et al. considered the energy functional defined on (r,s)-tensorial bundles on M which is a particular case of the vector bundles considered here. The characterising condition of critical points for the energy functional on the space of sections of sphere bundles was shown in [29]. This gives rise to the notion of *harmonic section of a sphere bundle*. Additionally, we analyse when such harmonic sections are also harmonic maps. In particular, we will show a method, mainly based on Lemma 3.3, for computing the tension field which is adequate for many situations. Concretely, when we consider Riemannian manifolds $(M, \langle\cdot,\cdot\rangle)$ equipped with some G-structure compatible with the metric $\langle\cdot,\cdot\rangle$. It is well known that, associated with many such G-structures, there are tensors Ψ of constant length stabilised by the action of the Lie group G. For many of those Ψ, our method proves to be efficient for computing their corresponding tension fields.

Some of the tensors playing the rôle of Ψ are: the Kähler form of an almost Hermitian structure, the fundamental three-form of a G_2-structure, the real and imaginary parts of the complex volume form of a special almost Hermitian structure, etc. So that we analyse the harmonicity of sections Ψ of sphere bundles both as sections and as maps. If the intrinsic torsion ξ^G of the G-structure vanishes, the harmonicity in both senses trivially follows. Therefore, we study examples defined on manifolds equipped with a G-structure such that $\xi^G \neq 0$, but with a geometry strongly conditioned by the G-structure. Thus, we have found examples of harmonic sections and harmonic maps into sphere bundles defined on manifolds with G-structure such that their respective intrinsic torsions have to be contained in a one-dimensional G-module (in many cases, this implies that the manifold is Einstein): nearly Kähler 6-manifolds, 7-manifolds with nearly parallel G_2-structure, Sasakian manifolds, Kenmotsu manifolds, etc.

Finally, we focus attention on manifolds equipped with a locally conformal parallel G-structure. The geometry of such manifolds is very conditioned by a closed one-form, called the *Lee form*. Thus, for such geometries, we have found tensor fields which are harmonic sections of sphere bundles. Furthermore, for some of them, if the Lee form is parallel, then they are also harmonic maps.

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2. Preliminaries

The *energy* of a map $f : (M, \langle \cdot, \cdot \rangle_M) \to (N, \langle \cdot, \cdot \rangle_N)$ between Riemannian manifolds, M being compact and oriented, is the integral

$$\mathcal{E}(f) = \frac{1}{2} \int_M \|f_*\|^2 dv, \tag{2.1}$$

where $\|f_*\|$ is the norm of the differential f_* of f with respect to the metrics $\langle \cdot, \cdot \rangle_M$, $\langle \cdot, \cdot \rangle_N$ and dv denotes the volume form on $(M, \langle \cdot, \cdot \rangle_M)$. On the domain of a local orthonormal frame field $\{e_1, \dots, e_n\}$ on M, $\dim M = n$, $\|f_*\|^2$ can be expressed as $\|f_*\|^2 = \langle f_* e_i, f_* e_i \rangle_N$, where the summation convention is used. Such a convention will be followed in the sequel. When a risk of confusion appear, the sum will be written in detail.

The critical points of the functional $\mathcal{E}$ on $C^\infty(M, N)$ are known as *harmonic maps* and, when M is closed, they have been characterised as maps with vanishing *tension field*. The tension of f is the vector field $\tau(f)$ along f which can be locally expressed as

$$\tau(f) = \widetilde{\nabla}_{e_i}(f_* e_i) - f_* \nabla_{e_i} e_i, \tag{2.2}$$

where $\widetilde{\nabla}$ denotes the *induced connection*, compatible with $\langle \cdot, \cdot \rangle_N$, on the *induced vector bundle* $f^*TN = \{(x, v) \,|\, x \in M,\ v \in T_{f(x)}N\}$ of TN by f. Here, the fibre on x in f^*TN is identified with $T_{f(x)}N$ and the Equation (2.2) is understood in this way. Then $\tau(f)$ is a smooth section of f^*TN. Denote by $\Gamma^\infty(f^*TN)$ the space of all smooth sections of f^*TN, also known as the space of all *variational vector fields* along f. Such a space can be regarded as the tangent space $T_f C^\infty(M, N)$ at f of the manifold $C^\infty(M, N)$.

If $\bar{N} \subset N$ is a regular submanifold of N such that $f(M) \subset \bar{N}$, then f belongs to $C^\infty(M, \bar{N})$ and the tangent projection $\tan(V)$ of $V \in \Gamma^\infty(f^*TN)$ to $\bar{N}$ is a vector field along $f : M \to \bar{N}$. Moreover, for $X \in \mathfrak{X}(M)$ and $\bar{V} \in \Gamma^\infty(f^*T\bar{N})$, we get $\tan(\widetilde{\nabla}^N_X \bar{V}) = \widetilde{\nabla}^{\bar{N}}_X \bar{V}$, where $\widetilde{\nabla}^N$ and $\widetilde{\nabla}^{\bar{N}}$ respectively denote the induced connections via $f : M \to N$ and via $f : M \to \bar{N}$. Hence, $\tan \tau(f)$ is the tension field of $f : M \to \bar{N}$ and we have

Lemma 2.1. *The map $f : (M, \langle \cdot, \cdot \rangle_M) \to (\bar{N}, \langle \cdot, \cdot \rangle_N)$ is harmonic if and only if the tension field of $f : (M, \langle \cdot, \cdot \rangle_M) \to (N, \langle \cdot, \cdot \rangle_N)$ is orthogonal to $\bar{N}$.*

3. Harmonicity of sections of sphere bundles

Let $\pi : \mathbb{E} \to M$ be a vector bundle over an n-dimensional Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ with typical fibre $\mathbb{E}_x \cong \mathbb{F}^m$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and denote by $L(\mathbb{E})$ the principal frame bundle of $\mathbb{E}$. A point p of $L(\mathbb{E})$ is a pair $(x; p_1, \dots, p_m)$ where $x \in M$ and $\{p_1, \dots, p_m\}$ is a basis of $\mathbb{E}_x$. We consider a *metric connection* $\Gamma : p \in L(\mathbb{E}) \mapsto \mathcal{H}_p \subset T_p L(\mathbb{E})$, with respect to a *fibre metric* on $\mathbb{E}$, which we will also denote by $\langle \cdot, \cdot \rangle$.

For each $v \in \mathbb{E}_x$, the corresponding horizontal subspace $\mathcal{H}_v$ of $T_v\mathbb{E}$ is defined as $\mathcal{H}_v = (\phi_\lambda)_{*p} \mathcal{H}_p$, where $(p, \lambda) \in L(\mathbb{E}) \times \mathbb{F}^m$ with $v = \sum_i \lambda^i p_i$ and ϕ_λ is the mapping $\phi_\lambda : L(\mathbb{E}) \to \mathbb{E}$ given by $\phi(q) = \sum_i \lambda^i q_i$, for all $q \in L(\mathbb{E})$. Hence, we obtain that $T_v\mathbb{E}$ is decomposed into $T_v\mathbb{E} = \mathcal{H}_v \oplus \mathcal{V}_v$, being $\mathcal{V}_v$ the vertical subspace $\mathcal{V}_v = T_v\mathbb{E}_x$, and $(M, \langle \cdot, \cdot \rangle)$ acquires a covariant derivative ∇ on the space of the smooth sections $\Gamma^\infty(\mathbb{E})$ on $\mathbb{E}$ by using of the notion of parallel displacement of fibres of $\mathbb{E}$. Because Γ is a metric connection, it follows that ∇ is *compatible*

with $\langle\cdot,\cdot\rangle$, that is, it satisfies

$$X\langle\sigma_1,\sigma_2\rangle = \langle\nabla_X\sigma_1,\sigma_2\rangle + \langle\sigma_1,\nabla_X\sigma_2\rangle, \tag{3.1}$$

for all vector field X on M and σ_1, $\sigma_2 \in \Gamma^\infty(\mathbb{E})$. Moreover, we define $\langle\nabla\sigma_1,\nabla\sigma_2\rangle$ in terms of local orthonormal tangent frames $\{e_1,\dots,e_n\}$ by the expression $\langle\nabla\sigma_1,\nabla\sigma_2\rangle = \langle\nabla_{e_i}\sigma_1,\nabla_{e_i}\sigma_2\rangle$.

The manifold $\mathbb{E}$ admits a Riemannian metric which generalises the *Sasaki metric* of the tangent bundle, see [1]. It will be denoted by $\langle\cdot,\cdot\rangle^S$ and, for $\xi_1,\xi_2 \in T_v\mathbb{E}$, it is defined by

$$\langle\xi_1,\xi_2\rangle^S = \langle\pi_*(\xi_1),\pi_*(\xi_2)\rangle + \langle K(\xi_1),K(\xi_2)\rangle, \tag{3.2}$$

where $K : T\mathbb{E} \to \mathbb{E}$ is the *connection map* [28] of the connection on $\mathbb{E}$ associated with Γ. We recall that such a map is given by $K(\xi) = \iota(\xi^{\mathsf{v}})$, where ξ^{v} is the vertical component of $\xi \in T\mathbb{E}$ and ι is the projection $\iota : T\mathbb{E} \to \mathbb{E}$ defined by $\iota(\eta) = 0$ for all $\eta \in \mathcal{H}$ and $\iota(u_v) = u$, for all $u_v \in \mathcal{V}_v$, being $u_v = \alpha'(0)$ and $\alpha(t) = v + tu$. In particular, if σ is a smooth section $\sigma \in \Gamma^\infty(\mathbb{E})$, one obtains that $K(\sigma_* X) = \nabla_X\sigma$, for any vector field $X \in \mathfrak{X}(M)$.

If M is compact and oriented, the energy functional of a smooth section $\sigma \in \Gamma^\infty(\mathbb{E})$ is defined as the energy of the map $\sigma : (M,\langle\cdot,\cdot\rangle) \to (\mathbb{E},\langle\cdot,\cdot\rangle^S)$. Then, from (2.1) and using (3.2), the energy $\mathcal{E}(\sigma)$ of σ can be expressed as

$$\mathcal{E}(\sigma) = \frac{n}{2}\mathrm{Vol}(M) + \frac{1}{2}\int_M \|\nabla\sigma\|^2 dv.$$

The relevant part of this formula, $B(\sigma) = \frac{1}{2}\int_M \|\nabla\sigma\|^2 dv$, is usually called the *total bending* of the section σ. It is immediate that $B : \Gamma^\infty(\mathbb{E}) \to \mathbb{R}$ is always non-negative and $B(\sigma)$ is zero if and only if σ is ∇-parallel. Thus, the notion of total bending provides a measure of how a section of $\pi : \mathbb{E} \to M$ fails to be parallel.

Each smooth section $\sigma \in \Gamma^\infty(\mathbb{E})$ determines a vertical vector field σ^{vert} on $\mathbb{E}$ given by $\sigma^{vert}_v = \sigma(x)_v \in \mathcal{V}_v$, for all $v \in \mathbb{E}_x$, and likewise, each vector field X on M can be lifted to a horizontal vector field X^{hor} on $\mathbb{E}$, its *horizontal lift*.

In the sequel, we will make use of the *musical isomorphisms* $\flat : TM \to T^*M$ and $\sharp : T^*M \to TM$, induced by the metric $\langle\cdot,\cdot\rangle$, respectively defined by $X^\flat = \langle X,\cdot\rangle$ and $\langle\theta^\sharp,\cdot\rangle = \theta$.

The tension field $\tau(\sigma)$ of σ has been characterised in [15] as follows

$$\tau(\sigma) = \left((R_{(\sigma,\langle\cdot,\cdot\rangle)})^\sharp\right)^{hor} \circ\sigma - (\nabla^*\nabla\sigma)^{vert}\circ\sigma,$$

where $R_{(\sigma,\langle\cdot,\cdot\rangle)}$ is the one-form on M given by

$$R_{(\sigma,\langle\cdot,\cdot\rangle)}(X) = \langle R_{X,e_i}\sigma,\nabla_{e_i}\sigma\rangle, \tag{3.3}$$

for any vector field X on M, where $\{e_1,\dots,e_n\}$ is an orthonormal frame field and $\nabla^*\nabla\sigma$ is the *connection Laplacian* (or *rough Laplacian*) [21] defined by

$$\nabla^*\nabla\sigma = -\left(\nabla^2\sigma\right)_{e_i,e_i}.$$

Here $R_{X,Y}\sigma = \nabla_{[X,Y]}\sigma - \nabla_X\nabla_Y\sigma + \nabla_Y\nabla_X\sigma$ and $(\nabla^2\sigma)_{X,Y} = \nabla_X\nabla_Y\sigma - \nabla_{(\nabla_X Y)}\sigma$. Hence, the map $\sigma : (M,\langle\cdot,\cdot\rangle) \to (\mathbb{E},\langle\cdot,\cdot\rangle^S)$ is harmonic if and only if $R_{(\sigma,\langle\cdot,\cdot\rangle)} = 0$ and $\nabla^*\nabla\sigma = 0$. Because for M compact, the connection Laplacian of σ vanishes if and only if σ is parallel (see [21, page 154]), it follows that σ is harmonic if and only if σ is parallel.

A critical point $\sigma \in \Gamma^\infty(\mathbb{E})$ of the restriction $\mathcal{E} : \Gamma^\infty(\mathbb{E}) \to I\!R$ of the energy functional to the space of sections is called a *harmonic section*. Consider $\sigma_t \in \Gamma^\infty(\mathbb{E})$ a smooth variation of σ

through sections. Then the corresponding *variation vector field* $x \in M \mapsto V(x) = \frac{d}{dt}_{|t=0}\sigma_t(x)$ is a section of the induced bundle $\sigma^*\mathcal{V}$ of the vertical subbundle $\sigma^*\mathcal{V} \subset T\mathbb{E}$. Using the first variation formula

$$\frac{d}{dt}_{|t=0}\mathcal{E}(\sigma_t) = -\int_M \langle V, \tau(\sigma)\rangle^S dv,$$

critical points $\sigma \in \Gamma^\infty(\mathbb{E})$ of the restriction of $\mathcal{E}$ to $\Gamma^\infty(\mathbb{E})$ are characterised by the vanishing of the vertical component of their tension. Hence, we can conclude

Proposition 3.1. *Let $\pi : \mathbb{E} \to M$ be a vector bundle with a metric connection over a closed and oriented Riemannian manifold and $\sigma \in \Gamma^\infty(\mathbb{E})$. Then the following statements are equivalent:*

(i) *the map $\sigma : (M, \langle\cdot,\cdot\rangle) \to (\mathbb{E}, \langle\cdot,\cdot\rangle^S)$ is harmonic;*
(ii) *σ is a harmonic section;*
(iii) *σ is parallel.*

Denote by $S_{\mathbb{E}}(r)$ the sphere bundle of radius $r > 0$ in $\mathbb{E}$ consisting of those elements $v \in \mathbb{E}$ with $\|v\| = r$. It is a subbundle and also a hypersurface of $\mathbb{E}$. For each $\sigma \in \Gamma^\infty(S_{\mathbb{E}}(r))$, $\frac{1}{r}\sigma^{vert} \circ \sigma$ is a unit normal vector field to $S_{\mathbb{E}}(r)$ along σ and the tangent projection $\tan \tau(\sigma)$ of $\tau(\sigma)$ to $S_{\mathbb{E}}(r)$ is given by

$$\begin{aligned}
\tan \tau(\sigma) &= \tau(\sigma) - \frac{1}{r^2}\langle \tau(\sigma), \sigma^{vert} \circ \sigma\rangle^S \sigma^{vert} \circ \sigma \\
&= \big((R_{(\sigma,\langle\cdot,\cdot\rangle)})^\sharp\big)^{hor} \circ \sigma - (\nabla^*\nabla\sigma)^{vert} \circ \sigma + \frac{1}{r^2}\langle (\nabla^*\nabla\sigma)^{vert} \circ \sigma, \sigma^{vert} \circ \sigma\rangle^S \sigma^{vert} \circ \sigma \\
&= \big((R_{(\sigma,\langle\cdot,\cdot\rangle)})^\sharp\big)^{hor} \circ \sigma + \Big(\frac{1}{r^2}\langle \nabla^*\nabla\sigma, \sigma\rangle\sigma - \nabla^*\nabla\sigma\Big)^{vert} \circ \sigma.
\end{aligned}$$

Hence, using Lemma 2.1, we have (see also [15])

Proposition 3.2. *Let $\pi : \mathbb{E} \to M$ be a vector bundle with a metric connection over a closed and oriented Riemannian manifold and $\sigma \in \Gamma^\infty(S_{\mathbb{E}}(r))$. Then, we have:*

(i) *the map $\sigma : (M, \langle\cdot,\cdot\rangle) \to (S_{\mathbb{E}}(r), \langle\cdot,\cdot\rangle^S)$ is harmonic if and only if $R_{(\sigma,\langle\cdot,\cdot\rangle)} = 0$ and $\nabla^*\nabla\sigma$ is collinear with σ.*
(ii) *σ is a critical point of $\mathcal{E}$ restricted to $\Gamma^\infty(S_{\mathbb{E}}(r))$ if and only if $\nabla^*\nabla\sigma$ is collinear with σ.*

For general Riemannian manifolds $(M, \langle\cdot,\cdot\rangle)$, not necessarily closed and oriented, a section of $S_{\mathbb{E}}(r)$ satisfying this last condition is called a *harmonic section of the sphere bundle* $S_{\mathbb{E}}(r)$. If a harmonic section σ is such that $R_{(\sigma,\langle\cdot,\cdot\rangle)} = 0$, then it is also a harmonic map into $(S_{\mathbb{E}}(r), \langle\cdot,\cdot\rangle^S)$. In such a case, we refer to σ as a *harmonic map into a sphere bundle.*

Let $(\nabla\sigma)^{\mathrm{t}} : \Gamma^\infty(\mathbb{E}) \to \mathfrak{X}(M)$ be the *transpose* operator of $\nabla\sigma$ with respect to $\langle\cdot,\cdot\rangle$ defined as

$$\langle (\nabla\sigma)^{\mathrm{t}}\varphi, X\rangle = \langle \varphi, \nabla_X\sigma\rangle, \quad \varphi \in \Gamma^\infty(\mathbb{E}),\ X \in \mathfrak{X}(M).$$

The following identity relating the connection Laplacian and the transpose operator is satisfied [21, page 155]

$$\langle \nabla^*\nabla\sigma, \varphi\rangle = -\mathrm{div}\,(\nabla\sigma)^{\mathrm{t}}\varphi + \langle \nabla\sigma, \nabla\varphi\rangle, \tag{3.4}$$

which motivates the notation chosen for the connection Laplacian.

If $\|\sigma\| = r$, then (3.1) implies that $(\nabla\sigma)^t\sigma = 0$. Therefore, using Equation (3.4), we have $\langle \nabla^*\nabla\sigma, \sigma\rangle = \|\nabla\sigma\|^2$. Hence, $\sigma \in \Gamma^\infty(S_{\mathbb{E}}(r))$ is a harmonic section of $S_{\mathbb{E}}(r)$ if and only if

$$\nabla^*\nabla\sigma = \frac{1}{r^2}\|\nabla\sigma\|^2\sigma. \tag{3.5}$$

The next result will be useful in the discussion of some examples to decide whether a harmonic section is a harmonic map.

Lemma 3.3. *Given a harmonic section σ of the sphere bundle $S_{\mathbb{E}}(r)$, the one-form $R_{(\sigma,\langle\cdot,\cdot\rangle)}$ defined in Equation (3.3) can be also written as*

$$R_{(\sigma,\langle\cdot,\cdot\rangle)}(X) = \operatorname{div}\left((\nabla\sigma)^{\mathrm{t}}\nabla_X\sigma\right) + \langle \nabla_{[X,e_i]}\sigma, \nabla_{e_i}\sigma\rangle - \tfrac{1}{2}X\left(\|\nabla\sigma\|^2\right). \tag{3.6}$$

Moreover, if $\langle \nabla_X\sigma, \nabla_Y\sigma\rangle$ is locally expressed by

$$\langle \nabla_X\sigma, \nabla_Y\sigma\rangle = \sum_{i=1}^n k_i e_i^\flat \otimes e_i^\flat(X,Y),$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame field and $k_1, \dots, k_n$ are smooth functions, then

$$R_{(\sigma,\langle\cdot,\cdot\rangle)} = \sum_{i=1}^n \{e_i(k_i) + \sum_{j=1}^n (k_i - k_j)\langle \nabla_{e_j}e_i, e_j\rangle\} e_i^\flat - \tfrac{1}{2}d(\sum_{j=1}^n k_j). \tag{3.7}$$

In particular, if $k_1 = \cdots = k_n = \lambda$, where λ is a (non-negative) constant, then σ is a harmonic map into $(S_{\mathbb{E}}(r), \langle\cdot,\cdot\rangle)$.

Proof. From (3.3), using the definition of the curvature operator, we have

$$R_{(\sigma,\langle\cdot,\cdot\rangle)}(X) = \langle (\nabla^2\sigma)_{e_i,X}, \nabla_{e_i}\sigma\rangle - \langle (\nabla^2\sigma)_{X,e_i}, \nabla_{e_i}\sigma\rangle.$$

From this identity, it is immediate to derive

$$R_{(\sigma,\langle\cdot,\cdot\rangle)}(X) = \langle \nabla_{e_i}(\nabla_X\sigma), \nabla_{e_i}\sigma\rangle + \langle \nabla_{[X,e_i]}\sigma, \nabla_{e_i}\sigma\rangle - \langle \nabla_X(\nabla_{e_i}\sigma), \nabla_{e_i}\sigma\rangle.$$

Now, $\langle \nabla_X(\nabla_{e_i}\sigma), \nabla_{e_i}\sigma\rangle = \frac{1}{2}X\left(\|\nabla\sigma\|^2\right)$ and, by Equation (3.4), we have

$$R_{(\sigma,\langle\cdot,\cdot\rangle)}(X) = \operatorname{div}\left((\nabla\sigma)^{\mathrm{t}}\nabla_X\sigma\right) + \langle \nabla^*\nabla\sigma, \nabla_X\sigma\rangle + \langle \nabla_{[X,e_i]}\sigma, \nabla_{e_i}\sigma\rangle - \tfrac{1}{2}X\left(\|\nabla\sigma\|^2\right).$$

Therefore, using (3.5), Equation (3.6) follows. To show Equation (3.7), we directly apply (3.6), taking into account that $(\nabla\sigma)^{\mathrm{t}}\nabla_{e_i}\sigma = k_i e_i$. □

Finally, we give the first and the second variation formula or the *Hessian form* of the energy functional $\mathcal{E}$ restricted to the set of all sections of the sphere bundle. Firstly, we note that $\Gamma^\infty(\mathbb{E})$ is a module over the ring of $\mathbb{F}$-valued functions and, for each $\sigma \in \Gamma^\infty(S_{\mathbb{E}}(r))$, one obtains the decomposition $\Gamma^\infty(\mathbb{E}) = \mathcal{V}(\sigma)^\perp \oplus \mathcal{V}(\sigma)$, where $\mathcal{V}(\sigma)$ is the submodule spanned by σ and $\mathcal{V}(\sigma)^\perp$ is the orthogonal complement to σ on $\Gamma^\infty(\mathbb{E})$, with respect to the metric fibre in $\mathbb{E}$. Then $\Gamma^\infty(S_{\mathbb{E}}(r))$ can be endowed with a structure of Fréchet manifold compatible with its C^∞-topology such that each $\sigma \in \Gamma^\infty(S_{\mathbb{E}}(r))$ is contained in a chart modelled on $\mathcal{V}(\sigma)^\perp$ and, consequently $T_\sigma\Gamma^\infty(S_{\mathbb{E}}(r)) = \mathcal{V}(\sigma)^\perp$. Moreover, a smooth variation σ_t, $t \in]-\varepsilon, \varepsilon[$, of σ through sections of $S_{\mathbb{E}}(r)$ can be regarded as a smooth curve $\gamma : t \mapsto \gamma(t) = \sigma_t$ in $\Gamma^\infty(S_{\mathbb{E}}(r))$ with $\gamma(0) = \sigma$ and $\gamma'(0) \in \mathcal{V}(\sigma)^\perp$.

Proposition 3.4. *Let $\pi : \mathbb{E} \to M$ be a vector bundle with a metric connection over a closed and oriented Riemannian manifold and let $\mathcal{E} : \Gamma^\infty(S_\mathbb{E}(r)) \to \mathbb{R}$ be the energy functional on $\Gamma^\infty(S_\mathbb{E}(k))$. We have*

(i) $d\mathcal{E}_\sigma(\varphi) = \int_M \langle \nabla^*\nabla\sigma, \varphi\rangle\, dv$, *for each* $\sigma \in \Gamma^\infty(S_\mathbb{E}(r))$ *and* $\varphi \in \mathcal{V}(\sigma)^\perp$.

(ii) *If σ is a harmonic section of $S_\mathbb{E}(r)$, then the Hessian form* $(\mathrm{Hess}\ \mathcal{E})_\sigma$ *on* $\mathcal{V}(\sigma)^\perp \cong T_\sigma\Gamma^\infty(S_\mathbb{E}(r))$ *is given by*

$$(\mathrm{Hess}\ \mathcal{E})_\sigma\varphi = \int_M (\|\nabla\varphi\|^2 - \|\varphi\|^2\|\nabla\sigma\|^2)dv.$$

Proof. If $\gamma :]-\varepsilon, \varepsilon[\to \Gamma^\infty(S_\mathbb{E}(r))$ is a curve such that $\gamma(0) = \sigma$, $\gamma'(0) = \varphi \in \mathcal{V}(\sigma)^\perp$, then we obtain

$$d\mathcal{E}_\sigma(\varphi) = \frac{d}{dt}_{|t=0}\mathcal{E}(\gamma(t)) = \frac{1}{2}\int_M \frac{d}{dt}_{|t=0}\|\nabla\gamma\|^2 dv = \int_M \langle\nabla\sigma, \nabla\varphi\rangle dv.$$

Now, using Equation (3.4) and taking into account that M is closed, we get (i).

For (ii),

$$\begin{aligned}(\mathrm{Hess}\ \mathcal{E})_\sigma\varphi &= \frac{d^2}{dt^2}_{|t=0}\mathcal{E}(\gamma(t)) = \frac{1}{2}\int_M \frac{d^2}{dt^2}_{|t=0}\|\nabla\gamma\|^2 dv \\ &= \int_M \frac{d}{dt}_{|t=0}\langle\nabla\gamma, \nabla\gamma'\rangle = \int_M (\|\nabla\varphi\|^2 + \langle\nabla\sigma, \nabla\gamma''(0)\rangle)dv.\end{aligned}$$

But, using Equation (3.4) as before, we get

$$\langle\nabla\sigma, \nabla\gamma''(0)\rangle = \mathrm{div}((\nabla\sigma)^{\mathrm{t}}\gamma''(0)) + \langle\gamma''(0), \nabla^*\nabla\sigma\rangle,$$

and therefore

$$(\mathrm{Hess}\ \mathcal{E})_\sigma\varphi = \int_M \left(\|\nabla\varphi\|^2 + \langle\gamma''(0), \nabla^*\nabla\sigma\rangle\right) dv.$$

Now, taking into account that $\|\gamma(t)\|^2 = r^2$ and σ is a harmonic section of the sphere bundle, we obtain

$$\langle\gamma''(0), \nabla^*\nabla\sigma\rangle = \langle\gamma''(0), \sigma\rangle\|\nabla\sigma\|^2 = -\|\varphi\|^2\|\nabla\sigma\|^2.$$

Hence (ii) follows. □

4. Differential forms as harmonic maps

Denote by $\bigwedge^p M$ the vector bundle of p-forms on M and by $\Omega^p M$ the space of its sections, that is, the space of differential p-forms on M. On $\bigwedge^p M$ we will consider the natural fibre metric $\langle\cdot,\cdot\rangle$ defined by

$$\langle\Psi, \Phi\rangle = \Psi(e_{i_1}, \ldots, e_{i_p})\Phi(e_{i_1}, \ldots, e_{i_p}), \tag{4.8}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame. Clearly, the covariant derivative ∇ on $\Omega^p M$ obtained as an extension of the Levi Civita connection associated to the metric $\langle\cdot,\cdot\rangle$ on M is compatible with such fibre metric, i.e., Equation (3.1) is satisfied.

Next Theorem is a first application of Lemma 3.3 and will be extremely useful in working with examples to be able to claim that certain harmonic sections of some particular sphere bundles are also harmonic maps.

Theorem 4.1. *Let $(M, \langle\cdot,\cdot\rangle)$ be an n-dimensional Riemannian manifold and (Ψ, Φ) a pair of differential forms of constant length $\|\Psi\| = r_1$ and $\|\Phi\| = r_2$, $\Psi \in \Omega^p M$ and $\Phi \in \Omega^{p+1} M$. If $\nabla_X \Psi = \lambda X \lrcorner \Phi$ and $\nabla_X \Phi = \mu X^\flat \wedge \Psi$, where λ, μ are constants and $0 \leq p < n$, then Ψ and Φ are harmonic maps into the corresponding sphere bundles $S_{\Omega^p M}(r_1)$ and $S_{\Omega^{p+1} M}(r_2)$.*

Proof. For $x \in M$, taking an orthonormal frame field $\{e_1, \dots, e_n\}$, such that $(\nabla_{e_i} e_j)_x = 0$, it follows that

$$\begin{aligned}(\nabla^*\nabla\Psi)_x &= -\nabla_{e_{ix}}(\nabla_{e_i}\Psi) = -\lambda\mu e_{ix}\lrcorner(e^\flat_{ix} \wedge \Psi_x) = -(n-p)\lambda\mu\Psi_x,\\ (\nabla^*\nabla\Phi)_x &= -\nabla_{e_{ix}}(\nabla_{e_i}\Phi) = -\lambda\mu e^\flat_{ix} \wedge (e_{ix}\lrcorner\Phi_x) = -(p+1)\lambda\mu\Phi_x.\end{aligned}$$

Hence, (Ψ, Φ) is a pair of harmonic sections of the respective sphere bundles and, using (3.5), we get

$$\|\nabla\Psi\|^2 = -(n-p)\lambda\mu r_1^2, \qquad \|\nabla\Phi\|^2 = -(p+1)\lambda\mu r_2^2.$$

Next, we compute $R_{(\Psi,\langle\cdot,\cdot\rangle)}$ and $R_{(\Phi,\langle\cdot,\cdot\rangle)}$. It is straightforwardly obtained

$$\operatorname{div}\Big((\nabla\Psi)^{\mathrm{t}}\nabla_X\Psi\Big)) = \tfrac{1}{2}(n-p+1)\lambda\mu X(\|\Psi\|^2) + \lambda^2\langle e_i\lrcorner\Phi, \nabla_{e_i}X\lrcorner\Phi\rangle = \lambda^2\langle e_i\lrcorner\Phi, \nabla_{e_i}X\lrcorner\Phi\rangle.$$

Also it is direct to obtain

$$\begin{aligned}\langle\nabla_{[X,\cdot]}\Psi, \nabla_\cdot\Psi\rangle =& \lambda^2\{\langle\nabla_X e_i\lrcorner\Phi, e_i\lrcorner\Phi\rangle - \langle e_i\lrcorner\Phi, \nabla_{e_i}X\lrcorner\Phi\rangle\}\\ =& -\lambda^2\{\tfrac{1}{2(p+1)}X(\|\Phi\|^2) + \langle e_i\lrcorner\Phi, \nabla_{e_i}X\lrcorner\Phi\rangle\}\\ =& -\lambda^2\langle e_i\lrcorner\Phi, \nabla_{e_i}X\lrcorner\Phi\rangle.\end{aligned}$$

Therefore, by Lemma 3.3, we have $R_{(\Psi,\langle\cdot,\cdot\rangle)} = 0$. We also obtain

$$\begin{aligned}\operatorname{div}\Big((\nabla\Phi)^{\mathrm{t}}\nabla_X\Phi\Big) =& \tfrac{1}{2}(p+1)\mu X(\lambda\|\Phi\|^2 + \mu\|\Psi\|^2) + \mu^2\langle e_i^\flat \wedge \Psi, (\nabla_{e_i}X)^\flat \wedge \Psi\rangle\\ =& \mu^2\langle e_i^\flat \wedge \Psi, (\nabla_{e_i}X)^\flat \wedge \Psi\rangle.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle\nabla_{[X,e_i]}\Phi, \nabla_{e_i}\Phi\rangle =& \mu^2\{\langle(\nabla_X e_i)^\flat \wedge \Psi, e_i^\flat \wedge \Psi\rangle - \langle(\nabla_{e_i}X)^\flat \wedge \Psi, e_i^\flat \wedge \Psi\rangle\}\\ =& \mu^2\{\tfrac{1}{2}(p+1)(2p+1-n)X(\|\Psi\|^2) - \langle e_i^\flat \wedge \Psi, (\nabla_{e_i}X)^\flat \wedge \Psi\rangle\}\\ =& -\mu^2\langle e_i^\flat \wedge \Psi, (\nabla_{e_i}X)^\flat \wedge \Psi\rangle.\end{aligned}$$

Now, using again Lemma 3.3, we have $R_{(\Phi,\langle\cdot,\cdot\rangle)} = 0$ and then (Ψ, Φ) is moreover a pair of harmonic maps into their respective sphere bundles. □

5. Examples of harmonic maps

First we recall some notions relative to G-structures, where G is a subgroup of the linear group $GL(n, \mathbb{R})$. An n-dimensional manifold M is equipped with a G-structure, if its frame bundle admits a reduction to the subgroup G. If M possesses a G-structure, then there always exists a G-connection defined on M. Moreover, if $(M, \langle\cdot,\cdot\rangle)$ is an oriented Riemannian n-manifold and G is a closed and connected subgroup of $SO(n)$, then there exists a unique metric G-connection $\nabla^G = \nabla + \xi^G$ such that $\xi^G \in T^*M \otimes \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ denotes the orthogonal complement in $\mathfrak{so}(n)$ of the Lie algebra $\mathfrak{g}$ of G and ∇ denotes the Levi-Civita connection [12].

The tensor ξ^G is called the *intrinsic torsion* of the G-structure and ∇^G is said to be the *minimal G-connection*.

5.1. **Nearly Kähler 6-manifolds.** An almost Hermitian manifold is a Riemannian $2n$-manifold $(M, \langle\cdot,\cdot\rangle)$ endowed with an almost complex structure J compatible with the metric. The presence of such a structure is equivalent to say that M is equipped with a $U(n)$-structure. Under the action of $U(n)$, the space $T^*M \otimes \mathfrak{u}(n)^\perp$ of possible intrinsic torsion tensors $\xi^{U(n)}$ is decomposed into irreducible $U(n)$-modules:

(1) if $n = 1$, $\xi^{U(1)} \in T^*M \otimes \mathfrak{u}(1)^\perp = \{0\}$;
(2) if $n = 2$, $\xi^{U(2)} \in T^*M \otimes \mathfrak{u}(2)^\perp = \mathcal{W}_2^{U(2)} + \mathcal{W}_4^{U(2)}$;
(3) if $n \geqslant 3$, $\xi^{U(n)} \in T^*M \otimes \mathfrak{u}(n)^\perp = \mathcal{W}_1^{U(n)} + \mathcal{W}_2^{U(n)} + \mathcal{W}_3^{U(n)} + \mathcal{W}_4^{U(n)}$.

where $\mathcal{W}_i^{U(n)}$ are the irreducible $U(n)$-modules given by Gray and Hervella [19]. The vanishing of $U(n)$-components of $\xi^{U(n)}$ gives rise to a natural classification of almost Hermitian manifolds. Associated with the almost Hermitian structure, it is usually considered the two form $\omega = \langle\cdot, J\cdot\rangle$, called the *Kähler form.* One can use the $U(n)$-isomorphism $\xi^{U(n)} \to -\xi^{U(n)}\omega = \nabla\omega$ to identify the intrinsic $U(n)$-torsion with $\nabla\omega$. Thus, Gray and Hervella showed conditions expressed by means of $\nabla\omega$ to characterise classes of almost Hermitian manifolds.

Each fibre T_xM of the tangent bundle can be consider as a complex vector space by defining $iv = Jv$. We will write $T_xM_{\mathbb{C}}$ when we are regarding T_xM as such a space. If, for all $x \in M$, there exists a complex volume n-form on $T_xM_{\mathbb{C}}$ defined by

$$\Psi_x = (\Psi_+)_x + i\,(\Psi_-)_x\,,$$

such that Ψ_+ and Ψ_- are real global differential n-forms on M compatible with the almost Hermitian structure, then M is said to be a *special almost Hermitian manifold* (see [27] for details). Such a fact is equivalent to say that M is equipped with an $SU(n)$-structure. For higher dimensions, $n \geq 4$, the space $T^*M \otimes \mathfrak{su}(n)^\perp$ is decomposed into five irreducible $SU(n)$-modules $\mathcal{W}_1^{SU(n)}, \dots, \mathcal{W}_5^{SU(n)}$. The first four modules are such that $\mathcal{W}_i^{SU(n)} = \mathcal{W}_i^{U(n)}$, $i = 1, \dots, 4$, and $\mathcal{W}_5^{SU(n)} \cong T^*M$.

For $n = 3$, the space $T^*M \otimes \mathfrak{su}(3)^\perp$ of intrinsic $SU(3)$-torsion tensors is decomposed into the following modules ([8, 27])

$$T^*M \otimes \mathfrak{su}(3)^\perp = \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3^{SU(3)} + \mathcal{W}_4^{SU(3)} + \mathcal{W}_5^{SU(3)},$$

where $\mathcal{W}_i^+ + \mathcal{W}_i^- = \mathcal{W}_i^{U(3)}$, $i = 1, 2$, $\mathcal{W}_j^{SU(3)} = \mathcal{W}_j^{U(3)}$, $j = 3, 4$, $\mathcal{W}_5^{SU(3)} \cong T^*M$, $\mathcal{W}_1^+ \cong \mathcal{W}_1^- \cong \mathbb{R}$ and $\mathcal{W}_2^+ \cong \mathcal{W}_2^- \cong \mathfrak{su}(3)$.

When $\xi^{U(n)} \in \mathcal{W}_1^{U(n)}$, the almost Hermitian manifold is called nearly Kähler. Gray [18] showed that any nearly Kähler and non-Kähler 6-manifold is Einstein. Furthermore, the Einstein constant ρ is positive. In this case, one can consider the 3-form Ψ_+ of type $(3, 0)$ such that $3w_1^+\Psi_+ = d\omega$, where $5\left(w_1^+\right)^2 = \rho$. Now we define

$$\Psi_- = -\Psi_+(J\cdot,\cdot,\cdot)$$

and fix $\Psi_+ + i\Psi_-$ as complex volume form, obtaining an $SU(3)$-structure (special almost Hermitian structure) on the manifold. Such an $SU(3)$-structure is of type $\mathcal{W}_1^+$ (in this case

the $\mathcal{W}_5^{SU(3)}$-part of the intrinsic torsion vanishes, see [27]). Thus, we will have a two-form ω and a three-form Ψ_+, such that

$$\nabla_X \omega = w_1^+ X \lrcorner \Psi_+, \qquad \nabla_X \Psi_+ = -w_1^+ X^\flat \wedge \omega, \qquad \|\omega\|^2 = 6, \qquad \|\Psi_+\|^2 = 24.$$

Hence, the conditions contained in Theorem 4.1 are satisfied. Then we get

$$\nabla^* \nabla \omega = 4 \left(w_1^+\right)^2 \omega, \quad \nabla^* \nabla \Psi_+ = 3 \left(w_1^+\right)^2 \Psi_+, \qquad R_{(\omega, \langle \cdot, \cdot \rangle)} = R_{(\Psi_+, \langle \cdot, \cdot \rangle)} = 0.$$

Additionally, we can also consider the pair consisting of the three-form Ψ_- and the four-form $\omega \wedge \omega$. Such forms satisfy

$$\begin{gathered} \nabla_X \Psi_- = \nabla_{JX} \Psi_+ = -w_1^+ J X^\flat \wedge \omega = \tfrac{1}{2} w_1^+ X \lrcorner (\omega \wedge \omega), \\ \nabla_X (\omega \wedge \omega) = 2 w_1^+ (X \lrcorner \Psi_+) \wedge \omega = -2 w_1^+ X^\flat \wedge \Psi_-, \\ \|\Psi_-\|^2 = 24, \qquad \|\omega \wedge \omega\|^2 = 144. \end{gathered}$$

Now, making use again of Theorem 4.1, we have

$$\nabla^* \nabla \Psi_- = 3 \left(w_1^+\right)^2 \Psi_-, \quad \nabla^* \nabla (\omega \wedge \omega) = 4 \left(w_1^+\right)^2 \omega \wedge \omega, \qquad R_{(\Psi_-, \langle \cdot, \cdot \rangle)} = R_{(\omega \wedge \omega, \langle \cdot, \cdot \rangle)} = 0.$$

In summary,

Theorem 5.1. *For a nearly Kähler 6-manifold, the differential forms ω, Ψ_+, Ψ_- and $\omega \wedge \omega$ are harmonic maps into their respective sphere bundles.*

5.2. **Nearly parallel G_2-manifolds.** A Riemannian seven-manifold M admits a G_2-structure if and only if there exists a three-form ϕ on M, nowhere zero, which is G_2-invariant and it is locally expressed by

$$\phi = \textstyle\sum_{i \in \mathbb{Z}_7} e_i^\flat \wedge e_{i+1}^\flat \wedge e_{i+3}^\flat,$$

where $\{e_0, \dots, e_6\}$ are certain local orthonormal frame fields. Such frames, usually called *Cayley frames*, are adapted to the G_2-structure and the seven-form $e_0^\flat \wedge \dots \wedge e_6^\flat = Vol$ is globally defined and fixed as volume form. Thus, corresponding to the volume form there is a Hodge star operator $*$. The four-form $*\phi$ is also G_2-invariant and locally expressed by

$$*\phi = -\textstyle\sum_{i \in \mathbb{Z}_7} e_{i+2}^\flat \wedge e_{i+4}^\flat \wedge e_{i+5}^\flat \wedge e_{i+6}^\flat.$$

Associated to the G_2-structure, we have the minimal G_2-connection $\nabla^{G_2} = \nabla + \xi^{G_2}$, such that $\xi^{G_2} \in T^*M \otimes \mathfrak{g}_2^\perp \subset T^*M \otimes \Lambda^2 T^*M$. In this case, the action of G_2 decomposes the space $T^*M \otimes \mathfrak{g}_2^\perp$ of possible intrinsic torsion tensors into four G_2-irreducible modules $\mathcal{W}_1^{G_2}, \dots, \mathcal{W}_4^{G_2}$ [14]. If one considers the G_2-modules of bilinear forms on $\mathbb{R}^7$ equipped with the usual Euclidean product $\langle \cdot, \cdot \rangle$, one can show that $\mathcal{W}_1^{G_2} \cong \mathbb{R}\langle \cdot, \cdot \rangle$, $\mathcal{W}_2^{G_2} \cong \mathfrak{g}_2$, $\mathcal{W}_3^{G_2} \cong S_0^2 \mathbb{R}^{7*}$, $\mathcal{W}_4^{G_2} \cong \mathfrak{g}_2^\perp \cong \mathbb{R}^7$. When $\xi^{G_2} \in \mathcal{W}_1^{G_2}$, the G_2-structure is called *nearly parallel.* In such a case, the manifold is Einstein [17], $\nabla_X \phi = \frac{k}{4} X \lrcorner * \phi$, $\nabla_X * \phi = -\frac{k}{4} X^\flat \wedge \phi$ and $\rho = \frac{k^2}{16}$ is the Einstein constant [24]. Since $4\|\phi\|^2 = \| * \phi\|^2 = 7.4!$, we are in the conditions of Theorem 4.1. Therefore,

$$\nabla^* \nabla \phi = \tfrac{k^2}{4} \phi = 4\rho\phi, \quad \nabla^* \nabla * \phi = \tfrac{k^2}{4} * \phi = 4\rho * \phi, \quad R_{(\phi, \langle \cdot, \cdot \rangle)} = R_{(*\phi, \langle \cdot, \cdot \rangle)} = 0.$$

In summary,

Theorem 5.2. *For a nearly parallel G_2-manifold, the differential forms ϕ and $*\phi$ are harmonic maps into their respective sphere bundles.*

5.3. **a-Sasakian manifolds.** An almost contact metric manifold is a Riemannian $(2n+1)$-manifold $(M, \langle\cdot,\cdot\rangle)$ equipped with a $(1,1)$-tensor φ and a one-form η such that $\langle \varphi X, \varphi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y)$ and $\varphi^2 = -I + \eta\otimes\zeta$, where $\zeta^\flat = \eta$. The presence of the mentioned tensors on the manifold is equivalent to say that M is endowed with a $U(n)\times 1$-structure, where $U(n)\times 1$ is considered such that $U(n)\times 1 \subseteq SO(2n+1)$. In this case, the cotangent space at each point T^*_xM is not irreducible under the action of the group $U(n)\times 1$. In fact, $T^*_xM = \mathbb{R}\eta + \eta^\perp$ and

$$\mathfrak{so}(2n+1) \cong \Lambda^2 T^*M \cong \Lambda^2\eta^\perp + \eta^\perp \wedge \mathbb{R}\eta = \mathfrak{u}(n) + \mathfrak{u}(n)^\perp_{|\zeta^\perp} + \eta^\perp\wedge\mathbb{R}\eta.$$

Therefore, for the space $T^*M\otimes\mathfrak{u}(n)^\perp$ of possible intrinsic $U(n)\times 1$-torsion, we have

$$T^*M\otimes\mathfrak{u}(n)^\perp = \eta^\perp\otimes\mathfrak{u}(n)^\perp_{|\zeta^\perp} \;+\; \eta\otimes\mathfrak{u}(n)^\perp_{|\zeta^\perp} \;+\; \eta^\perp\otimes\eta^\perp\wedge\eta \;+\; \eta\otimes\eta^\perp\wedge\eta.$$

Chinea and González-Dávila [7] showed that $T^*M\otimes\mathfrak{u}(n)^\perp$ is decomposed into twelve irreducible $U(n)$-modules $\mathcal{C}_1,\dots,\mathcal{C}_{12}$, where

$$\begin{aligned}
\eta^\perp\otimes\mathfrak{u}(n)^\perp_{|\zeta^\perp} &= \mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3+\mathcal{C}_4,\\
\eta^\perp\otimes\eta^\perp\wedge\eta &= \mathcal{C}_5+\mathcal{C}_8+\mathcal{C}_9+\mathcal{C}_6+\mathcal{C}_7+\mathcal{C}_{10},\\
\eta\otimes\mathfrak{u}(n)^\perp_{|\zeta^\perp} &= \mathcal{C}_{11},\\
\eta\otimes\eta^\perp\wedge\eta &= \mathcal{C}_{12}.
\end{aligned}$$

Note that $\mathcal{C}_1,\dots,\mathcal{C}_4$ are the Gray and Hervella's $U(n)$-modules, i.e., $\mathcal{C}_i \cong \mathcal{W}_i^{U(n)}$. Furthermore, note that φ restricted to $\zeta^\perp$ works as an almost complex structure and, if one considers the $U(n)$-action on the bilinear forms $\bigotimes^2\eta^\perp$, then we have the decomposition

$$\bigotimes^2\eta^\perp = \mathbb{R}\langle\cdot,\cdot\rangle_{|\zeta^\perp} + \mathfrak{su}(n)_s + \sigma^{2,0} + \mathbb{R}F + \mathfrak{su}(n)_a + \mathfrak{u}(n)^\perp_{\zeta|^\perp},$$

where F is the form, called the *fundamental two-form*, defined by $F = \langle\cdot,\varphi\cdot\rangle$.

The modules $\mathfrak{su}(n)_s$ (resp., $\mathfrak{su}(n)_a$) consist of Hermitian symmetric (resp., skew-symmetric) bilinear forms orthogonal to $\langle\cdot,\cdot\rangle_{|\zeta^\perp}$ (resp., F) and $[\![\sigma^{2,0}]\!]$ $(\mathfrak{u}(n)^\perp_{|\zeta^\perp})$ is the space of anti-Hermitian symmetric (resp., skew-symmetric) bilinear forms. With respect to the modules $\mathcal{C}_i$, one has $\eta^\perp\otimes\eta^\perp\wedge\mathbb{R}\eta \cong \bigotimes^2\eta^\perp$ and, using the $U(n)$-map $\xi^{U(n)} \to -\xi^{U(n)}\eta = \nabla\eta$, it is obtained

$$\mathcal{C}_5 \cong \mathbb{R}\langle\cdot,\cdot\rangle_{|\zeta^\perp},\quad \mathcal{C}_8 \cong \mathfrak{su}(n)_s,\quad \mathcal{C}_9 \cong [\![\sigma^{2,0}]\!],\quad \mathcal{C}_6 \cong \mathbb{R}F,\quad \mathcal{C}_7 \cong \mathfrak{su}(n)_a,\quad \mathcal{C}_{10} \cong \mathfrak{u}(n)^\perp_{|\zeta^\perp}.$$

In summary, under the action of $U(n)\times 1$, the space of possible intrinsic torsion tensors $\mathrm{T}^*M\otimes\mathfrak{u}(n)^\perp$ is decomposed into:

(1) if $n=1$, $\xi^{U(1)} \in \mathrm{T}^*M\otimes\mathfrak{u}(1)^\perp = \mathcal{C}_5+\mathcal{C}_6+\mathcal{C}_9+\mathcal{C}_{12}$;
(2) if $n=2$, $\xi^{U(2)} \in \mathrm{T}^*M\otimes\mathfrak{u}(2)^\perp = \mathcal{C}_2+\mathcal{C}_4+\cdots+\mathcal{C}_{12}$;
(3) if $n\geqslant 3$, $\xi^{U(n)} \in \mathrm{T}^*M\otimes\mathfrak{u}(n)^\perp = \mathcal{C}_1+\cdots+\mathcal{C}_{12}$.

Here, we will consider the particular case that the intrinsic $U(n)\times 1$-torsion is contained in $\mathcal{C}_6$. In such a case, M is called an a-Sasakian manifold (Sasakian when a=1) and it is characterised by the condition

$$\nabla_X F = -aX^\flat\wedge\eta.$$

Note that $d^*F(\zeta) = 2na$, where d^* means the coderivative. Moreover, if $n \geq 2$, then a is constant [23].

Since for almost contact metric manifolds one has $(\nabla_X \eta)\, Y = (\nabla_X F)\,(\zeta, \varphi Y)$, then, for an a-Sasakian manifold, it follows $\nabla_X \eta = a X \lrcorner F$.

Now, we consider a-Sasakian $(2n+1)$-manifolds, $n \geq 2$, and the pairs of differential forms $(\eta \wedge F^r, F^{r+1})$, where $0 \leq r \leq n$ and $F^r = F \wedge \overset{(r)}{\dots} \wedge F$. Such pairs satisfy

$$\nabla_X (\eta \wedge F^r) = \tfrac{1}{r+1}\, a\, X \lrcorner F^{r+1}, \qquad \nabla_X F^{r+1} = -(r+1) a\, X^\flat \wedge \eta \wedge F^r.$$

Thus, they satisfy the conditions required in Theorem 4.1. Note that $\|\eta \wedge F^r\|^2 = \frac{(2r+1)!r!n!}{(n-r)!}$ and $\|F^{r+1}\|^2 = \frac{(2(r+1))!(r+1)!n!}{(n-r-1)!}$. Therefore, we get

$$\nabla^*\nabla\,(\eta \wedge F^r) = 2(n-r)a^2 \eta \wedge F^r, \quad \nabla^*\nabla F^{r+1} = 2(r+1)a^2 F^{r+1},$$
$$R_{(\eta \wedge F^r, \langle \cdot,\cdot \rangle)} = R_{(F^{r+1}, \langle \cdot,\cdot \rangle)} = 0.$$

In summary,

Theorem 5.3. *For an a-Sasakian $(2n+1)$-manifold, the differential forms $\eta \wedge F^r$ and F^{r+1}, $0 \leq r \leq n$, are harmonic maps into their respective sphere bundles.*

Note that, in general, a-Sasakian manifolds are not Einstein. Diverse interesting aspects of the so-called Sasakian-Einstein manifolds can be found in [4,5].

5.4. **3-a-Sasakian manifolds.** A $(4n+3)$-manifolds M is said to be endowed with an *almost contact metric 3-structure*, if M has a Riemannian metric $\langle \cdot, \cdot \rangle$ and three almost contact metric structures $(\varphi_i, \zeta_i, \eta_i)$, $i = 1, 2, 3$, satisfying

$$\eta_i(\zeta_j) = \delta_{ij}, \quad \varphi_i(\zeta_j) = -\varphi_j(\zeta_i) = \zeta_k, \quad \eta_i \circ \varphi_j = -\eta_j \circ \varphi_i = \eta_k,$$
$$\varphi_i \circ \varphi_j - \eta_j \otimes \zeta_i = -\varphi_j \circ \varphi_i - \eta_i \otimes \zeta_j = \varphi_k, \qquad \langle \varphi_i X, \varphi_i Y \rangle = \langle X, Y \rangle - \eta_i(X)\eta_i(Y),$$

for all cyclic permutation (ijk) of (123). In the language of G-structures, one says that the manifold is equipped with an $Sp(n) \times I_3$-structure. Here, it is considered $Sp(n) \times I_3 \subseteq SO(4n) \times I_3 \subseteq SO(4n+3)$.

Now, we will consider $(4n+3)$-manifolds endowed with an almost contact metric 3-structure such that the three structures are a-Sasakian. If a_i is the constant corresponding to the structure i, $i = 1, 2, 3$, it is immediate to show that $a_1 = a_2 = a_3 = a$. In this case, the manifold is called a *3-a-Sasakian manifold* (*3-Sasakian*, when $a = 1$). Kashiwada [22] proved that 3-Sasakian manifolds are Einstein, being $\rho = 2(2n+1)$ the Einstein constant.

On 3-a-Sasakian manifolds, we consider the pair of differential forms $(\Psi^{(r)}, \Omega^{(r)})$, where

$$\Psi^{(r)} = \textstyle\sum_{i=1}^3 \eta_i \wedge F_i^r, \qquad \Omega^{(r)} = \textstyle\sum_{i=1}^3 F_i^{r+1}.$$

The tensors Ω and Ψ are of constant length and

$$\nabla_X \Psi^{(r)} = -\tfrac{a}{r+1} X \lrcorner \Omega^{(r)}, \qquad \nabla_X \Omega^{(r)} = (r+1) a X \wedge \Psi^{(r)}.$$

Now, making use of Theorem 4.1, we have

$$\nabla^*\nabla \Psi^{(r)} = 2(2n+1-r)a^2 \Psi^{(r)}, \quad \nabla^*\nabla \Omega^{(r)} = 2(r+1)a^2 \Omega^{(r)}, \quad R_{(\Psi^{(r)}, \langle \cdot,\cdot \rangle)} = R_{(\Omega^{(r)}, \langle \cdot,\cdot \rangle)} = 0.$$

Another pair of differential forms to be considered consists of $\circlearrowright\!\!\bigoplus_{ijk}^{123} \eta_i \wedge F_j \wedge F_k$ and $F_1 \wedge F_2 \wedge F_3$, where $\circlearrowright\!\!\bigoplus$ denotes cyclic sum over the listed elements. For such forms, we have

$$\nabla_X \left(\circlearrowright\!\!\bigoplus_{ijk}^{123} \eta_i \wedge F_j \wedge F_k \right) = -aX\lrcorner\, (F_1 \wedge F_2 \wedge F_3)$$

and

$$\nabla_X (F_1 \wedge F_2 \wedge F_3) = aX \wedge \left(\circlearrowright\!\!\bigoplus_{ijk}^{123} \eta_i \wedge F_j \wedge F_k \right).$$

Since these forms are of constant length, we can use Theorem 4.1. Therefore,

$$\nabla^*\nabla \left(\circlearrowright\!\!\bigoplus_{ijk}^{123} \eta_i \wedge F_j \wedge F_k \right) = 2(2n-1)a^2 \left(\circlearrowright\!\!\bigoplus_{ijk}^{123} \eta_i \wedge F_j \wedge F_k \right),$$

$$\nabla^*\nabla (F_1 \wedge F_2 \wedge F_3) = 6a^2 F_1 \wedge F_2 \wedge F_3,$$

$$R_{(\circlearrowright\!\!\bigoplus_{ijk}^{123} \eta_i \wedge F_j \wedge F_k, \langle\cdot,\cdot\rangle)} = R_{(F_1\wedge F_2 \wedge F_3, \langle\cdot,\cdot\rangle)} = 0.$$

Now, we consider the pair $\eta_i \wedge F_j + \eta_j \wedge F_i$ and $F_i \wedge F_j$, where $i \neq j$. Since these forms are of constant length and

$$\nabla_X (\eta_i \wedge F_j + \eta_j \wedge F_i) = -aX\lrcorner(F_i \wedge F_j), \qquad \nabla_X (F_i \wedge F_j) = aX \wedge (\eta_i \wedge F_j + \eta_j \wedge F_i),$$

then, by Theorem 4.1,

$$\nabla^*\nabla (\eta_i \wedge F_j + \eta_j \wedge F_i) = 4na^2 (\eta_i \wedge F_j + \eta_j \wedge F_i), \qquad \nabla^*\nabla(F_i \wedge F_j) = 4a^2 F_i \wedge F_j,$$

$$R_{(\eta_i\wedge F_j + \eta_j \wedge F_i, \langle\cdot,\cdot\rangle)} = R_{(F_i \wedge F_j, \langle\cdot,\cdot\rangle)} = 0.$$

In summary,

Theorem 5.4. *For a 3-a-Sasakian $(4n+3)$-manifold, the differential forms $\sum_{i=1}^3 \eta_i \wedge F_i^r$, $\sum_{i=1}^3 F_i^{r+1}$, $\circlearrowright\!\!\bigoplus_{ijk}^{123} \eta_i \wedge F_j \wedge F_k$, $F_1 \wedge F_2 \wedge F_3$, $\eta_i \wedge F_j + \eta_j \wedge F_i$ and $F_i \wedge F_j$, where $0 \leq r \leq 2n+1$ and (ijk) is a cyclic permutation of (123), are harmonic maps into their respective sphere bundles.*

Finally, for 3-a-Sasakian manifolds, we will also find some differential forms which are eigenvectors with respect to the connection Laplacian and they do not follow the scheme contained in Theorem 4.1. In fact, we will discuss the harmonicity as a map of such forms by applying Lemma 3.3. For instance, let us consider the three-form

$$\vartheta = (2n+3)\eta_1 \wedge \eta_2 \wedge \eta_3 + \textstyle\sum_{i=1}^3 \eta_i \wedge F_i.$$

If $\{e_1, \ldots, e_{4n}, e_{4n+1}, e_{4n+2}, e_{4n+3}\}$ is a local orthonormal frame field such that at $x \in M$ is the basis for vectors $\{e_{1\,x}, \ldots, e_{4n\,x}, e_{4n+1\,x} = \zeta_{1\,x}, e_{4n+2\,x} = \zeta_{2\,x}, e_{4n+3\,x} = \zeta_{3\,x}\}$ adapted to the 3-Sasakian structure and $(\nabla_{e_i} e_j)_x = 0$, then

$$\nabla^*\nabla \eta_1 \wedge \eta_2 \wedge \eta_3 = 12(n+1)a^2 \eta_1 \wedge \eta_2 \wedge \eta_3 + 4a^2 \textstyle\sum_{i=1}^3 \eta_i \wedge F_i.$$

Now, from this last identity, $\nabla^*\nabla\Psi^{(1)} = 4n\Psi^{(1)}$ and using the identities

$$\textstyle\sum_{r=1}^{4n+3} (e_r\lrcorner(e_r \wedge \eta_i) = 2(2n+1)\eta_i, \qquad \sum_{r=1}^{4n+3} (e_r\lrcorner F_i) \wedge (e_r\lrcorner F_j) = -2F_k - \eta_i \wedge \eta_j, \tag{5.9}$$

a straightforward computation shows that $\nabla^*\nabla\vartheta = 12(n+1)a^2\vartheta$.

Now we compute $R_{(\vartheta,\langle\cdot,\cdot\rangle)}$. One can deduce that ϑ is of constant length and

$$\langle \nabla_X \vartheta, \nabla_Y \vartheta \rangle = 18(4n^2+10n+3)a^2 \langle X, Y\rangle - 6(12n^2+26n+9)a^2 \textstyle\sum_{i=1}^3 \eta_i(X)\eta_i(Y).$$

Now, making use of Lemma 3.3, we have

$$R_{(\vartheta,\langle\cdot,\cdot\rangle)} = 6(12n^2+26n+9)a^2 \sum_{i=1}^{4n}\sum_{j=1}^{3}\left(\eta_j(\nabla_{\zeta_j}e_i)e_i^\flat - \langle\nabla_{e_i}\zeta_j, e_i\rangle\eta_j\right).$$

But, for 3-a-Sasakian manifolds, one obtains

$$\nabla\zeta_i = a\sum_{s=1}^{n}\left(\varphi_i e_s^\flat\otimes e_s - e_s^\flat\otimes\varphi_i e_s + \varphi_j e_s^\flat\otimes\varphi_k e_s - \varphi_k e_s^\flat\otimes\varphi_j e_s\right) + a(\eta_k\otimes\zeta_j - \eta_j\otimes\zeta_k),$$

where (ijk) is a cyclic permutation of (123). Therefore, for $i = 1,\dots,4n$ and $j = 1,2,3$, it follows

$$\eta_j(\nabla_{\zeta_j}e_i) = \langle\nabla_{e_i}\zeta_j, e_i\rangle = 0, \tag{5.10}$$

and, as a consequence, $R_{(\vartheta,\langle\cdot,\cdot\rangle)} = 0$.

In the same context of 3-a-Sasakian manifolds, we have

$$\begin{aligned}\nabla_X\left(\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge\eta_j\wedge F_k\right) &= a\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge(X\lrcorner F_j)\wedge F_k - a\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge F_j\wedge(X\lrcorner F_k)\\ &\quad -3aX^\flat\wedge\eta_1\wedge\eta_2\wedge\eta_3.\end{aligned}$$

From this identity we get

$$\begin{aligned}\nabla_{e_{s\,p}}\left(\nabla_{e_s}\left(\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge\eta_j\wedge F_k\right)\right) &= 2a^2\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}(e_s\lrcorner F_i)\wedge(e_s\lrcorner F_j)\wedge F_k\\ &\quad -2a^2\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge e_s\lrcorner(e_s\wedge\eta_j)\wedge F_k\\ &\quad -5a^2\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge\eta_j\wedge e_s\wedge(e_s\lrcorner F_k).\end{aligned}$$

Thus, taking Equations (5.9) into account, we obtain

$$\nabla^*\nabla\left(\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge\eta_j\wedge F_k\right) = 8(n+2)a^2\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge\eta_j\wedge F_k + 4a^2\Omega^{(1)}.$$

Now, since $\nabla^*\nabla\Omega^{(1)} = 4\Omega^{(1)}$, it follows

$$\nabla^*\nabla\left(\Omega^{(1)} + (2n+3)\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge\eta_j\wedge F_k\right) = 8(n+2)a^2\left(\Omega^{(1)} + (2n+3)\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge\eta_j\wedge F_k\right).$$

Writing $\Psi = \Omega^{(1)} + (2n+3)\circlearrowright\!\!\!\!\bigoplus_{ijk}^{123}\eta_i\wedge\eta_j\wedge F_k$, it is straightforward to check

$$\langle\nabla_X\Psi, \nabla_Y\Psi\rangle = 6(4n+1)a^2\langle X,Y\rangle - 6a^2\sum_{i=1}^{3}\eta_i(X)\eta_i(Y).$$

Now, making use of Lemma 3.3, we have

$$R_{(\Psi,\langle\cdot,\cdot\rangle)} = 6a^2\sum_{i=1}^{4n}\sum_{j=1}^{3}\left(\eta_j(\nabla_{\zeta_j}e_i)e_i^\flat - \langle\nabla_{e_i}\zeta_j, e_i\rangle\eta_j\right).$$

Again, using Equation (5.10) and the fact that Ψ is of constant length, we obtain $R_{(\Psi,\langle\cdot,\cdot\rangle)} = 0$.

Finally, since for 3-a-Sasakian manifolds we have

$$\begin{aligned}\nabla_X(\eta_i\wedge\eta_j) &= a(X\lrcorner F_i)\wedge\eta_j + \eta_i\wedge(X\lrcorner F_j),\\ \nabla_{e_{s\,p}}(\nabla_{e_s}(\eta_i\wedge\eta_j)) &= a^2(2(e_s\lrcorner F_i)\wedge(e_s\lrcorner F_j) - 2\eta_i\wedge\eta_j + \eta_i(e_s)e_s^\flat\wedge\eta_j + \eta_j(e_s)\eta_i\wedge e_s^\flat),\end{aligned}$$

we obtain $\nabla^*\nabla(\eta_i\wedge\eta_j) = 2a^2(2F_k + (4n+3)\eta_i\wedge\eta_j)$. As a consequence, for any (ijk) cyclic permutation, we have

$$\nabla^*\nabla(F_k + (2n+1)\eta_i\wedge\eta_j) = 2(4n+3)a^2(F_k + (2n+1)\eta_i\wedge\eta_j).$$

Moreover, it is direct to check that

$$\begin{aligned}\langle \nabla_X(F_k + (2n+1)\eta_i \wedge \eta_j), \nabla_Y(F_k + (2n+1)\eta_i \wedge \eta_j)\rangle =& 2(2(2n+1)^2+1)a^2\langle X, Y\rangle \\ &- 2(2n^2+2n+1)a^2 \textstyle\sum_{i=1}^3 \eta_i(X)\eta_i(Y).\end{aligned}$$

Therefore, using Lemma 3.3 and Equation (5.10), we get

$$R_{(F_k+(2n+1)\eta_i\wedge\eta_j,\langle\cdot,\cdot\rangle)} = 2(2n^2+2n+1)a^2 \textstyle\sum_{i=1}^{4n}\sum_{j=1}^{3}\left(\eta_j(\nabla_{\zeta_j}e_i)e_i^\flat - \langle \nabla_{e_i}\zeta_j, e_i\rangle \eta_j\right) = 0.$$

In summary,

Theorem 5.5. *For a 3-a-Sasakian $(4n+3)$-manifold, the differential forms*

$$\begin{aligned}&(2n+3)\eta_1\wedge\eta_2\wedge\eta_3 + \textstyle\sum_{i=1}^3 \eta_i \wedge F_i,\\ &\textstyle\sum_{i=1}^3 F_i \wedge F_i + (2n+3)\ \circlearrowright\!\!\!\!\bigoplus_{ijk}^{123} \eta_i\wedge\eta_j\wedge F_k,\\ &F_k + (2n+1)\eta_i\wedge\eta_j,\end{aligned}$$

where (ijk) is a cyclic permutation of (123), are harmonic maps into their respective sphere bundles.

5.5. **b-Kenmotsu manifolds.** Next we will consider almost contact metric manifolds M such that the intrinsic $U(n)\times 1$-torsion is contained in $\mathcal{C}_5$. In such a case, M is known as a *b-Kenmotsu manifold* and it is characterised by the condition

$$\nabla_X F = b\eta \wedge (X \lrcorner F).$$

Note that $\nabla_X\eta = -bX^\flat + b\eta(X)\eta$ and $\operatorname{div}\zeta = -d^*\eta = -2nb$, where d^* stands for the coderivative as we have pointed out above. The exterior derivative will be denoted by d. For b-Kenmotsu manifolds, η is closed and b is a function such that $db = f\eta$ [23].

Proposition 5.6. *For b-Kenmotsu manifolds, we have:*

$$\begin{aligned}\nabla^*\nabla(F^r) &= 2rb^2F^r,\\ R_{(F^r,\langle\cdot,\cdot\rangle)} &= 2rb\|F^r\|^2(rb^2-f)\eta,\\ \nabla^*\nabla(\eta\wedge F^r) &= 2(n-r)b^2\eta\wedge F^r,\\ R_{(\eta\wedge F^r,\langle\cdot,\cdot\rangle)} &= 2(2r+1)(n-r)\|F^r\|^2 b(b^2-f)\eta,\end{aligned}$$

where $0\le r\le n$. Moreover, we also have:

- (a) $R_{(F^r,\langle\cdot,\cdot\rangle)} = 0$ *if and only if one of the following equivalent conditions is satisfied:*
  - (i) $db(\zeta) = rb^2$,
  - (ii) $2n\,\operatorname{grad}(\operatorname{div}(\zeta)) = -r\operatorname{div}^2(\zeta)\zeta$,
  - (iii) $2n\,\Delta\eta = r(d^*\eta)^2\eta$;
- (b) $R_{(\eta\wedge F^r,\langle\cdot,\cdot\rangle)} = 0$ *if and only if one of the following equivalent conditions is satisfied:*
  - (i) $db(\zeta) = b^2$,
  - (ii) $2n\,\operatorname{grad}(\operatorname{div}(\zeta)) = -\operatorname{div}^2(\zeta)\zeta$,
  - (iii) $2n\,\Delta\eta = (d^*\eta)^2\eta$;

where Δ denotes the Hodge Laplacian, $\Delta = dd^* + d^*d$.

Note that $R_{(F^r,\langle\cdot,\cdot\rangle)}$ and $R_{(\eta\wedge F^r,\langle\cdot,\cdot\rangle)}$ are always closed and, in particular, if $\operatorname{div}\zeta = -d^*\eta = -2nb$ is a non-zero constant, then $R_{(F^r,\langle\cdot,\cdot\rangle)}$ and $R_{(\eta\wedge F^r,\langle\cdot,\cdot\rangle)}$ are nowhere zero.

Proof. By straightforward computation we obtain the expressions for $\nabla^*\nabla(F^r)$ and $\nabla^*\nabla(\eta\wedge F^r)$. Furthermore, from the identities

$$\langle\nabla_X F^r, \nabla_Y F^r\rangle = \tfrac{r^2}{n} b^2\|F^r\|^2(\langle X, Y\rangle - \eta(X)\eta(Y))$$

and

$$\langle\nabla_X(\eta\wedge F^r), \nabla_Y(\eta\wedge F^r)\rangle = \tfrac{(2r+1)(n-r)}{n} b^2\|F^r\|^2(\langle X, Y\rangle - \eta(X)\eta(Y)),$$

using Lemma 3.3, the expressions for $R_{(F^r,\langle\cdot,\cdot\rangle)}$ and $R_{(\eta\wedge F^r,\langle\cdot,\cdot\rangle)}$ follow. From these last expressions the remaining parts of Proposition are immediately deduced. □

Example 5.7. The one-form η is closed for b-Kenmotsu manifolds. This implies that one has local coordinates $(x_1,\dots,x_{2n},x_{2n+1})$ such that $dx_{2n+1}=\eta$. If k is a constant, we have:

(a) F^r is a harmonic map if and only if

$$b(x_1,\dots,x_{2n},x_{2n+1}) = -\frac{1}{r(x_{2n+1}+k)},$$

(b) $\eta\wedge F^r$ is a harmonic map if and only if

$$b(x_1,\dots,x_{2n},x_{2n+1}) = -\frac{1}{x_{2n+1}+k}.$$

In [23], Marrero considers the product manifold $M = L\times V$, where L is S^1 or an open interval, and $(V, J, \langle\cdot,\cdot\rangle_V)$ is a Kähler $2n$-manifold. Let E be a nowhere vanishing vector field on L and σ a positive function on L. Taking

$$\varphi(cE, X) = (0, JX), \quad \zeta = (E, 0), \quad \langle(cE, X), (dE, Y)\rangle = cd + \sigma\langle X, Y\rangle_V,$$

where c,d are functions on L and X, Y are vector fields on V, a b-Kenmotsu structure is obtained, $b = -\frac{1}{2}d(\ln\sigma)(E)$. In particular, we consider L as an open interval and $E = \frac{\partial}{\partial t}$, where t is the coordinate. For K and $C\neq 0$ constants, two interesting particular cases are:

(a)′ If we choose $\sigma = C(t+K)^{\frac{2}{r}}$, we will obtain that F^r is a harmonic map into sphere bundle. In fact, one checks that $b = -\frac{1}{r(t+K)}$ and $db(\zeta) = rb^2 = \frac{1}{r(t+K)^2}$.

(b)′ If we choose $\sigma = C(t+K)^2$, we will obtain that $\eta\wedge F^r$ is a harmonic map into a sphere bundle. In fact, one obtains $b = -\frac{1}{t+K}$ and $db(\zeta) = b^2 = \frac{1}{(t+K)^2}$.

5.6. **Locally conformal parallel p-forms.** Throughout this subsection, we are assuming that Ψ is a p-form of constant length.

Definition 5.8. A p-form Ψ on a Riemannian n-manifold M is said to be *locally conformal parallel*, if there exists a closed one-form θ on M such that

$$\nabla_X\Psi = X^\flat\wedge(\theta^\sharp\lrcorner\Psi) - \theta\wedge(X\lrcorner\Psi), \tag{5.11}$$

for all $X\in\mathfrak{X}(M)$. We will refer to the one-form θ as the Lee form of Ψ.

The following results will be useful in examples.

Proposition 5.9. *If Ψ is a locally conformal parallel p-form on a Riemannian n-manifold M with Lee form θ, then its coderivative $d^*\Psi$ and its rough Laplacian $\nabla^*\nabla\Psi$ are respectively given by*

$$d^*\Psi = (p-n)\theta^\sharp \lrcorner \Psi,$$
$$\nabla^*\nabla\Psi = p\|\theta\|^2\Psi + (n-2p)\theta \wedge (\theta^\sharp \lrcorner \Psi).$$

In particular, if $2p = n$, then Ψ is a harmonic section of its corresponding sphere bundle.

Proof. The expression for $d^*\Psi$ is obtained by a direct computation. In order to compute $(\nabla^*\nabla\Psi)_m$, for $m \in M$, we will consider a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $(\nabla_{e_i} e_j)_m = 0$. Thus, because Ψ is locally conformal parallel, we have

$$(\nabla^*\nabla\Psi)_m = -e_i^\flat \wedge (\theta^\sharp \lrcorner (\nabla_{e_i}\Psi)) - e_i^\flat \wedge ((\nabla_{e_i}\theta)^\sharp \lrcorner \Psi) + \nabla_{e_i}\theta \wedge (e_i \lrcorner \Psi) - \theta \wedge d^*\Psi.$$

Now, using the expression for $\nabla\Psi$ given in (5.11) and the identities $e_i \wedge (e_i \lrcorner \Psi) = p\Psi$ and $e_i^\flat \wedge \theta \wedge (\theta^\sharp \lrcorner (e_i \lrcorner \Psi)) = (p-1)\theta \wedge (\theta^\sharp \lrcorner \Psi)$, we obtain

$$-e_i^\flat \wedge (\theta^\sharp \lrcorner (\nabla_{e_i}\Psi)) = -p\theta \wedge (\theta^\sharp \lrcorner \Psi) + p\|\theta\|^2\Psi.$$

Moreover, because θ is closed, we have $(\nabla_X\theta)(Y) = (\nabla_Y\theta)(X)$ and it is not hard to see that

$$e_i^\flat \wedge ((\nabla_{e_i}\theta)^\sharp \lrcorner \Psi) = \nabla_{e_i}\theta \wedge (e_i \lrcorner \Psi).$$

Finally, from all this, the required expression for $\nabla^*\nabla\Psi$ follows. $\square$

In the examples below, in order to apply Lemma 3.3, we will need to compute $\langle \nabla_X\Psi, \nabla_Y\Psi\rangle$. For doing this we will use (5.11) to obtaine the expression

$$\begin{aligned} \langle \nabla_X\Psi, \nabla_Y\Psi\rangle &= p\|\theta^\sharp \lrcorner \Psi\|^2\langle X, Y\rangle + p\|\theta\|^2\langle X \lrcorner \Psi, Y \lrcorner \Psi\rangle \\ &\quad -p\theta(X)\langle \theta^\sharp \lrcorner \Psi, Y \lrcorner \Psi\rangle - p\theta(Y)\langle \theta^\sharp \lrcorner \Psi, X \lrcorner \Psi\rangle \\ &\quad -2p(p-1)\langle X \lrcorner (\theta^\sharp \lrcorner \Psi), Y \lrcorner (\theta^\sharp \lrcorner \Psi)\rangle. \end{aligned} \tag{5.12}$$

Example 5.10 (Locally conformal Kähler manifolds)**.** An almost Hermitian $2n$-manifold $(M, \langle\cdot,\cdot\rangle, J)$ of type $\mathcal{W}_4$ is characterised by the condition that its Kähler two-form is given by

$$\nabla_X\omega = X^\flat \wedge (\theta^\sharp \lrcorner \omega) - \theta \wedge (X \lrcorner \omega).$$

From this identity and the nature of ω, one can deduce that θ has to be closed. Thus, ω is a locally conformal parallel two-form.

Proposition 5.11. *If the Lee form θ of ω is not zero somewhere and $r < n$, then ω^r is a harmonic section of its corresponding sphere bundle if and only if $2r = n$.*

Proof. Direct computations show that

$$\|\omega^r\|^2 = \tfrac{(2r)!r!n!}{(n-r)!}, \qquad \|X \lrcorner \omega^r\|^2 = \tfrac{(2r)!r!n!}{2n(n-r)!}\|X\|^2. \tag{5.13}$$

Moreover, since $\nabla_X\omega^r = X^\flat \wedge (\theta^\sharp \lrcorner \omega^r) - \theta \wedge (X \lrcorner \omega^r)$, it follows from Proposition 5.9 that

$$\nabla^*\nabla\omega^r = 2r\|\theta\|^2\omega^r + 2(n-2r)\theta \wedge (\theta^\sharp \lrcorner \omega^r).$$

Additionally, if $\nabla^*\nabla\omega^r = f\omega^r$, taking into account that $\langle X^\flat \wedge \beta, \gamma\rangle = p\langle \beta, X \lrcorner \gamma\rangle$ into account for any $(p-1)$-form β and p-form γ, we have

$$f\|\omega^r\|^2 = 2r\|\theta\|^2\|\omega^r\|^2 + 4r(n-2r)\|\theta^\sharp \lrcorner \omega^r\|^2.$$

Now, making use of (5.13), we obtain

$$f = \tfrac{4r(n-r)}{n}\|\theta\|^2. \tag{5.14}$$

On the other hand, if we consider a local adapted orthonormal frame $\{e_1, Je_1, \dots, e_n, Je_n\}$ such that $\theta = \|\theta\| e_1$, by computing $\nabla^*\nabla\omega^r(e_1, Je_1, \dots, e_r, Je_r)$, we have

$$2r\|\theta\|^2 r! + 2(n-2r)\|\theta\|^2 r! = f r!.$$

Therefore, $f = 2(n-r)\|\theta\|^2$. From this and (5.14), it follows $n = 2r$. ◻

In order to compute $\langle \nabla_X\omega^r, \nabla_Y\omega^r\rangle$, a straightforward computation shows that

$$\begin{aligned}\langle X \lrcorner(\theta\lrcorner\omega^r), Y\lrcorner(\theta\lrcorner\omega^r)\rangle &= \tfrac{r r!(2r-2)!(n-2)!}{(n-r)!}\left\{(r-1)\|\theta\|^2\langle X, Y\rangle\right.\\ &\quad \left.-(r-1)\theta(X)\theta(Y) + (n-r)J\theta(X)J\theta(Y)\right\}.\end{aligned}$$

From this, (5.12) and (5.13), we obtain

$$\langle \nabla_X\omega^r, \nabla_Y\omega^r\rangle = \tfrac{2r\, r!(2r)!(n-2)!}{(n-r-1)!}(\|\theta\|^2\langle X, Y\rangle - \theta(X)\theta(Y) - J\theta(X)J\theta(Y)).$$

Finally, choosing a local orthonormal frame field as above such that $\theta = \|\theta\| e_1$, we will have $\|\theta\|\mathrm{div}(e_1) = -d^*\theta - d(\|\theta\|)(e_1)$ and $\|\theta\|\mathrm{div}(Je_1) = -d^*(J\theta) - d(\|\theta\|)(Je_1)$. Moreover, since θ is closed, then $\nabla_{\theta^\sharp}\theta = \|\theta\| d(\|\theta\|)$. Taking all this into account and the fact that $2r = n$ as well, and making use of Lemma 3.3, we will obtain

$$R_{(\omega^r, \langle\cdot,\cdot\rangle)} = \tfrac{1}{2}(n-1)(n!)^2\left(-(n-2)d(\|\theta\|^2) + d^*\theta\,\theta + d^*(J\theta)\,J\theta + \nabla_{J\theta^\sharp}J\theta\right).$$

In particular, if the Lee form θ is parallel, then it is not hard to see that $\nabla_{J\theta^\sharp}J\theta = 0$ and $d^*(J\theta) = d^*\omega(\theta^\sharp) = 2(n-1)J\theta(\theta^\sharp) = 0$. Therefore, $R_{(\omega^r, \langle\cdot,\cdot\rangle)} = 0$.

Theorem 5.12. *For a locally conformal Kähler $4n$-manifold, ω^n is a harmonic map into its sphere bundle if and only if*

$$(n-2)d(\|\theta\|^2) = (d^*\theta)\,\theta + d^*(J\theta)\,J\theta + \nabla_{J\theta^\sharp}J\theta,$$

where θ is the Lee form of ω. In particular, if θ is parallel, then ω^n is a harmonic map into its sphere bundle.

Hopf manifolds are diffeomorphic to $S^1 \times S^{2m-1}$ and admit a locally conformal Kähler structure with parallel Lee form θ [32]. Furthermore, θ is nowhere zero. In particular, if $m = 2n$, then $S^1 \times S^{4n-1}$ satisfies the conditions of last Theorem. In general, if we consider a Sasakian $(4n-1)$-manifold M, then the product manifold $\mathbb{R} \times M$ can be equipped with a locally conformal Kähler structure satisfying the conditions of Theorem 5.12.

Example 5.13 (Locally conformal parallel *Spin*(7)-manifolds)**.** Now, let us consider $\mathbb{R}^8$ endowed with an orientation and its standard inner product. Let $\{e, e_0, ..., e_6\}$ be an oriented orthonormal basis. Consider the four-form Φ on $\mathbb{R}^8$ given by

$$\Phi = \sum_{i\in\mathbb{Z}_7} e\wedge e_i \wedge e_{i+1}\wedge e_{i+3} - \sigma\sum_{i\in\mathbb{Z}_7} e_{i+2}\wedge e_{i+4}\wedge e_{i+5}\wedge e_{i+6}, \tag{5.15}$$

where σ is a fixed constant such that $\sigma = +1$ or $\sigma = -1$, and $+$ in the subindexes means the sum in $\mathbb{Z}_7$. We fix $e\wedge e_0\wedge\cdots\wedge e_6 = \frac{\sigma}{14}\Phi\wedge\Phi$ as a volume form.

The subgroup of $GL(8,\mathbb{R})$ which fixes Φ is isomorphic to the double covering *Spin*(7) of *SO*(7) [20]. Moreover, *Spin*(7) is a compact simply-connected Lie group of dimension 21 [6].

The Lie algebra $\mathfrak{spin}(7)$ of *Spin*(7) is isomorphic to the skew-symmetric two-forms ψ satisfying the linear equations

$$\sigma\psi(e_i, e) + \psi(e_{i+1}, e_{i+3}) + \psi(e_{i+4}, e_{i+5}) + \psi(e_{i+2}, e_{i+6}) = 0,$$

for all $i \in \mathbb{Z}_7$. Shortly, $\mathfrak{spin}(7) \cong \{\psi \in \Lambda^2 T * M | *(\psi \wedge \Phi) = \psi\}$, where $*$ is the Hodge star operator. The orthogonal complement $\mathfrak{spin}(7)^\perp$ of $\mathfrak{spin}(7)$ in $\Lambda^2\mathbb{R}^{8*} = \mathfrak{so}(8)$ is the seven-dimensional space generated by

$$\beta_i = \sigma e_i \wedge e + e_{i+1} \wedge e_{i+3} + e_{i+4} \wedge e_{i+5} + e_{i+2} \wedge e_{i+6}, \tag{5.16}$$

where $i \in \mathbb{Z}_7$. Equivalently, $\mathfrak{spin}(7)^\perp$ is described as the space consisting of those skew-symmetric two-forms ψ such that $*(\psi \wedge \Phi) = -3\psi$.

A *Spin*(7)*-structure* on an eight-manifold M^8 is by definition a reduction of the structure group of the frame bundle to *Spin*(7); we shall also say that M is a *Spin*(7)*-manifold*. This can be geometrically described by saying that there exists a nowhere vanishing global differential four-form Φ on M^8 and a local frame $\{e, e_0, \ldots, e_6\}$ such that the four-form Φ can be locally written as in (5.15). The four-form Φ is called the *fundamental form* of the *Spin*(7)-manifold M [2] and the local frame $\{e, e_0, \ldots, e_6\}$ is called *a Cayley frame*.

The fundamental form of a *Spin*(7)-manifold determines a Riemannian metric $\langle \cdot, \cdot \rangle$ through $\langle X, Y \rangle = -\frac{1}{7} * ((X \lrcorner \Phi) \wedge *(Y \lrcorner \Phi))$ [16]. Thus, $\langle \cdot, \cdot \rangle$ is called the metric induced by Φ. Any Cayley frame becomes an orthonormal frame with respect to such a metric.

There are four classes of *Spin*(7)-manifolds given by Fernández in [13]. They are obtained as irreducible *Spin*(7)-representations of the space $\overline{W} \cong \mathbb{R}^{8*} \otimes \mathfrak{spin}(7)^\perp$ of all possible covariant derivatives $\nabla\Phi$. The Lee form θ is defined by

$$\theta = -\tfrac{1}{7} * (*d\Phi \wedge \Phi) = \tfrac{1}{7} * (d^*\Phi \wedge \Phi). \tag{5.17}$$

Alternatively, the classification can be described in terms of the Lee form as follows : $\overline{W}_0$: $d\Phi = 0$; $\overline{W}_1 : \theta = 0$; $\overline{W}_2 : d\Phi = \theta \wedge \Phi$; $\overline{W} : \overline{W} = \overline{W}_1 + \overline{W}_2$ [25].

A *Spin*(7)-structure of the class $\overline{W}_2$ is called a *locally conformal parallel Spin*(7)*-structure*. This is motivated by the fact that the Lee form of a *Spin*(7)-structure in the class $\overline{W}_2$ is closed. Therefore, such a manifold is locally conformal to a parallel *Spin*(7)-manifold. Furthermore, the covariant derivative $\nabla\Phi$ for such manifolds is given by

$$4\nabla_X\Phi = X^\flat \wedge (\theta^\sharp \lrcorner \Phi) - \theta \wedge (X \lrcorner \Phi). \tag{5.18}$$

Theorem 5.14. *The fundamental form Φ of a locally conformal parallel Spin(7)-structure is a harmonic section of its corresponding sphere bundle. Furthermore, if θ denotes the Lee form of the Spin(7)-structure, then Φ is a harmonic map into its sphere bundle if and only if $(d^*\theta)\,\theta = 3d\left(\|\theta\|^2\right)$.*

In particular, since the *Spin*(7)-structure defined on the product of spheres $S^1 \times S^7$ in [25] is locally conformal parallel being the Lee form parallel, the corresponding four-form Φ on $S^1 \times S^7$ is a harmonic map into its sphere bundle.

Proof of Theorem 5.14. From Proposition 5.9 and Equation (5.18), we have $\nabla^*\nabla\Phi = \frac{1}{4}\|\theta\|^2\Phi$. Additionally, straightforward computations show that

$$\begin{aligned} \langle X \lrcorner \Phi, Y \lrcorner \Phi \rangle &= 42\langle X, Y \rangle, \\ \langle X \lrcorner (\theta^\sharp \lrcorner \Phi), Y \lrcorner (\theta^\sharp \lrcorner \Phi) \rangle &= 6\left(\|\theta\|^2\langle X, Y \rangle - \theta(X)\theta(Y)\right). \end{aligned}$$

Now, using these identities in Equation (5.12), we obtain

$$\langle \nabla_X \Phi, \nabla_Y \Phi \rangle = 12 \left(\|\theta\|^2 \langle X, Y \rangle - \theta(X)\theta(Y) \right).$$

Finally, making use of Lemma 3.3, we get

$$R_{(\Phi, \langle \cdot,\cdot \rangle)} = 12 \left((d^*\theta)\, \theta - 3d(\|\theta\|^2) \right).$$

This concludes the proof. □

Example 5.15 (Locally conformal quaternion-Kähler manifolds). A $4n$-dimensional manifold M is said to be *almost quaternion-Hermitian*, if M is equipped with an $Sp(n)\, Sp(1)$-structure. This is equivalent to the presence of a Riemannian metric $\langle \cdot, \cdot \rangle$ and a rank-three subbundle $\mathcal{G}$ of the endomorphism bundle $\operatorname{End} TM$, such that locally $\mathcal{G}$ has an *adapted basis* I, J, K satisfying $I^2 = J^2 = -1$ and $K = IJ = -JI$, and $\langle AX, AY \rangle = \langle X, Y \rangle$, for all $X, Y \in T_xM$ and $A = I, J, K$. An almost quaternion-Hermitian manifold with a global adapted basis is called an *almost hyperhermitian* manifold. In such a case the manifold is equipped with an $Sp(n)$-structure.

One may define a global, non-degenerate four-form Ω, the *fundamental form*, by the local formula

$$\Omega = \sum_{A=I,J,K} \omega_A \wedge \omega_A, \tag{5.19}$$

where $\omega_A(X, Y) = \langle X, AY \rangle$, $A = I, J, K$, are the three local Kähler-forms corresponding to an adapted basis.

To describe the intrinsic torsion of almost quaternion-Hermitian manifolds, it is usually used the E-H-formalism of [26, 30]. Thus, E is the fundamental representation of $Sp(n)$ on $\mathbb{C}^{2n} \cong \mathbb{H}^n$ via left multiplication by quaternionic matrices, considered in $GL(2n, \mathbb{C})$, and H is the representation of $Sp(1)$ on $\mathbb{C}^2 \cong \mathbb{H}$ given by $q.\zeta = \zeta\overline{q}$, for $q \in Sp(1)$ and $\zeta \in \mathbb{H}$. An $Sp(n)\, Sp(1)$-structure on a manifold M gives rise to local bundles E and H associated to these representations and identifies $TM \otimes_{\mathbb{R}} \mathbb{C} \cong E \otimes_{\mathbb{C}} H$.

The intrinsic $Sp(n)\, Sp(1)$-torsion ξ, $n > 1$, is in $T^*M \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp$. The decomposition of the space of possible intrinsic torsion tensors into irreducible $Sp(n)\, Sp(1)$-modules was obtained by Swann in [30] and is given by

$$T^*M \otimes (\mathfrak{sp}(n) + \mathfrak{sp}(1))^\perp = \Lambda^3_0 E S^3 H + K S^3 H + E S^3 H + \Lambda^3_0 E H + K H + E H, \tag{5.20}$$

where $\Lambda^3_0 E$ and K are certain irreducible $Sp(n)$-modules and $S^3 H$ means the symmetric 3-power of H (see [26] for details). If the dimension of M is at least 12, all the modules of the sum are non-zero. For an eight-dimensional manifold M, we have $\Lambda^3_0 E S^3 H = \Lambda^3_0 E H = \{0\}$. Therefore, for $\dim M \geq 12$ and $\dim M = 8$, we have respectively $2^6 = 64$ and $2^4 = 16$ classes of almost quaternion-Hermitian manifolds. In [26], by the identification $\xi \to -\xi\Omega = \nabla\Omega$, explicit conditions characterising these classes were given. So that they are expressed in terms of $\nabla\Omega$. However, from such conditions, it is not hard to derive descriptions for the $Sp(n)\, Sp(1)$-components of ξ.

A *locally conformal quaternion-Kähler $4n$-manifold* is an almost quaternion Hermitian manifold such that its intrinsic torsion ξ is in EH. This is equivalent to say that

$$\nabla_X \Omega = X^\flat \wedge (\theta^\sharp \lrcorner \Omega) - \theta \wedge (X \lrcorner \Omega),$$

where the Lee form θ is given by

$$\theta = \frac{1}{4(n-1)(2n+1)} * (*d\Omega \wedge \Omega).$$

Because of the nature of Ω, from $d\Omega = 4\theta \wedge \Omega$, one deduces that θ has to be closed. From Proposition 5.9 the next result follows.

Theorem 5.16. *For a locally conformal quaternion-Kähler $8n$-manifold, Ω^n is a harmonic section of its corresponding sphere bundle.*

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